The Optimum Inverse Problem of Numerical Error Analysis

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Numerical Problems

$$\left.\begin{array}{r}\text{inputs}\\ \text{initial data}\\ y\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{outputs}\\ \text{solution}\\ x\end{array}\right.$$

- Equations

$$F(y, x) = 0$$

F is the residual function.

- Solution function

$$f(y) = x$$

Two Perturbational Questions

$$F(y_0,\ x_0) = 0$$

$$F(\underbrace{y_0 + \Delta y}_{y'},\ \underbrace{x_0 + \Delta x}_{\bar{x}}) = 0$$

1. Is the solution mathematically stable?

$$\|\Delta x\| \le \chi\,\|\Delta y\| + o(\|\Delta y\|)$$

$$\chi_{\min} = \|\mathcal{D}f(x_0)\|$$

2. <u>Inverse</u> <u>Problem</u>. What size of Δy is needed to accomodate an $\bar{x}$?

Two Solutions of the <u>Inverse</u> <u>Problem</u>

A. *A priori* inverse rounding error analysis:
for any $\bar{x}$, construct a Δy and bound it.

$$||\Delta y|| \le \text{bound on backward errors}$$

B. <u>*A posteriori* analysis</u>

$$\mu(\bar{x}) = \left\{ \begin{array}{c} \text{minimal} \\ \text{optimal} \\ \text{smallest} \end{array} \right\} \begin{array}{l} \text{size of} \\ \text{backward} \\ \text{errors} \end{array}$$

$$= \min_{\{y' \,:\, F(y', \bar{x}) = 0\}} ||y' - y_0||$$

Linear Equations, $A_0 x = b_0$ [Oettli and Prager, 64]

$$\min_{\Delta A, \Delta b} \max_{i,j,k} \left\{ \left| \frac{\Delta A_{i,j}}{E_{i,j}} \right|, \left| \frac{\Delta b_k}{f_k} \right| \right\} = \max_j \left| \frac{(A_0\, \bar{x} - b_0)_j}{(|E|\, |\bar{x}| + |f|)_j} \right|$$

Linear Equations, $A_0 x = b$ [Rigal and Gaches, 67]

$$\min_{\Delta A} \|\Delta A\| = \frac{\|\bar{r}\|}{\|\bar{x}\|} \quad \text{where } \bar{r} = A_0 \bar{x} - b$$

Linear Least Squares [Waldén, Karlson, Sun, 95]

$$\min_{\Delta A} \|\Delta A\|_F = \sqrt{\frac{\|\bar{r}\|_2^2}{\|\bar{x}\|_2^2} + \min_i \left\{ 0,\ \lambda_i \left(A_0 A_0^t - \frac{\bar{r}\, \bar{r}^{\,t}}{\|\bar{x}\|_2^2} \right) \right\}}$$

Minimal Backward Error Bibliography

linear equations

various factorizations

linear least squares

invariant subspaces

1964	Oettli, Prager	structured LE
1967	Rigal, Gaches	LE
1989	Bunch, Demmel, Van Loan	symmetric LE
1991	Sun	Choleski and QR
1992	Bartels, Higham	Vandermonde LE
	Higham, Higham	Toeplitz LE
	Higham, Higham	multiple right side LE
1993	Sun	characteristic subspaces
1994	Chandrasekaran, Ipsen	sym. eigen decomp.
	Varah	Toeplitz LE
1995	Smoktunowicz	symmetric structured LE
	Smoktunowicz	eigenvalue and vector
	Sun	sym. eigen decomp.
	Waldén, Karlson, Sun	LLS
1996	Higham	alt. expression for LLS
	Sun	multiple right side LLS
1997	Karlson, Waldén	estimate for LLS
	Sun, Sun	underdetermined LE
	Sun	min. norm sltn. for LLS
1998	Frayssé, Toumazou	eigenvalue and vector
	Higham, Higham	eigenvalue and vector
	Sun	Vandermonde LE
1999	Cox, Higham	linearly constrained LLS
	Gu	estimate for LLS
2001	Malyshev	spherically constrained LLS
2002	Malyshev, Sadkane	evaluation for sparse LLS
	Stewart	Krylov subspaces

1. Accuracy criterion. [von Neumann, 47]
   If the initial data have error $\geq \mu(\bar{x})$, then $\bar{x}$ solves the problem, to the extent the problem is known.

2. Backward stability estimation.
   An $\bar{x}$ with a small $\mu(\bar{x})$ is backward stable.

3. Test new algorithms.
   Explore backward stability without having to do an inverse rounding error analysis.

Summary of Pertubation Analyses in NA

1. Forward Type	2. Inverse Type	
	A. Inverse Rounding Error Analysis	**B. Mathematical Analysis**
Find $\chi(y_0)$***, the condition number.*** $\chi_{\min} = \|\mathcal{D}f(y_0)\|$	***Construct some backward errors.***	***Find*** $\mu(\bar{x})$***, the minimal size of backward error.*** $\mu(\bar{x}) = ?$

Outline

Background

Abstract Formulation of the Problem

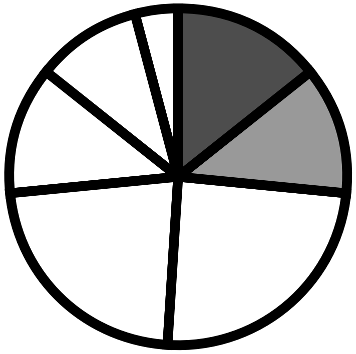
Perturbation Theory of Metric Projections

Asymptotic Approximation

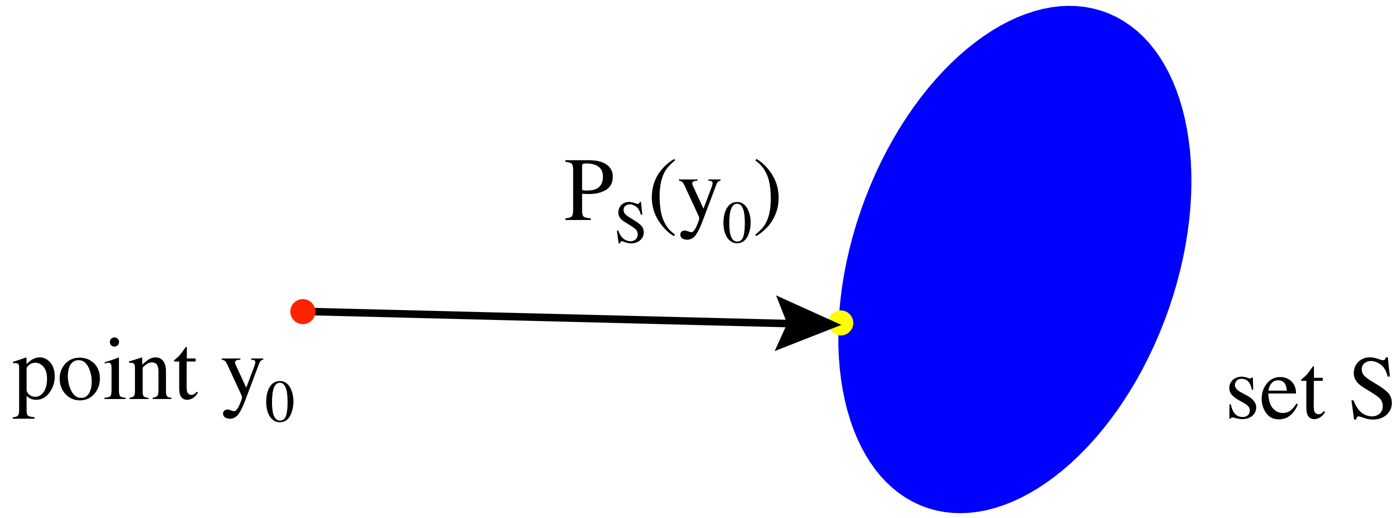
Application to Linear Least Squares

Numerical Examples

Conclusion



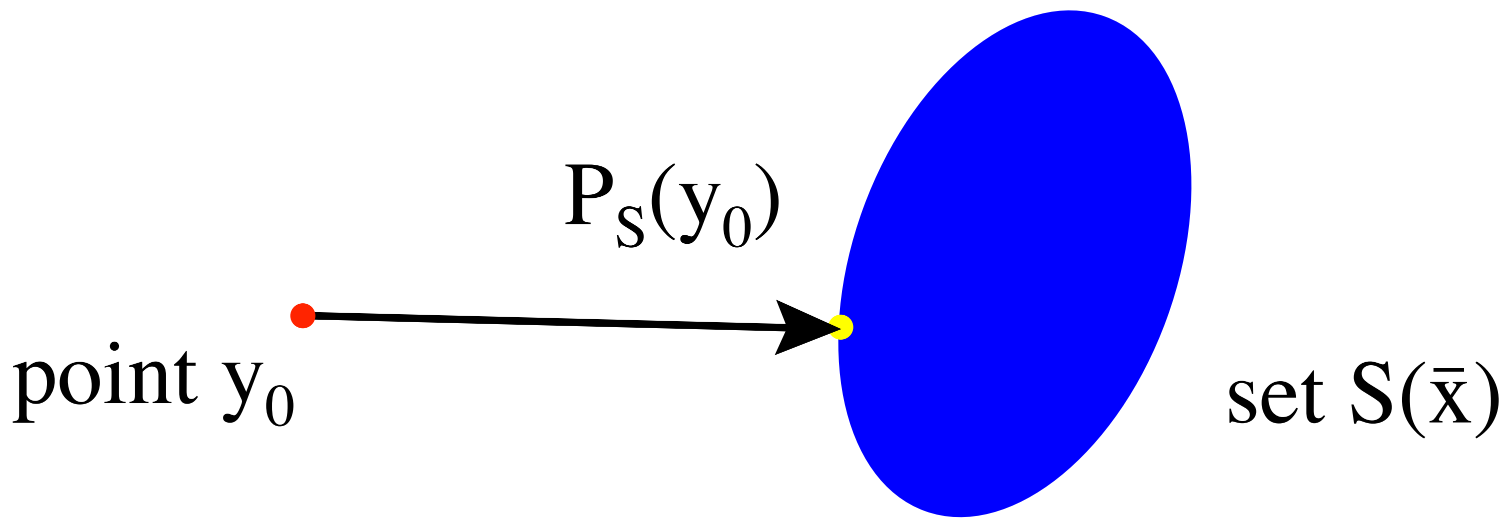
Metric Projection



$$\boldsymbol{P_S(y_0)} = \text{a point in set } \boldsymbol{S} \text{ nearest to } \boldsymbol{y_0}$$

$$\mathrm{dist}(\boldsymbol{y_0}, \boldsymbol{S}) = \text{distance from } \boldsymbol{y_0} \text{ to } \boldsymbol{S}$$

Metric Projection

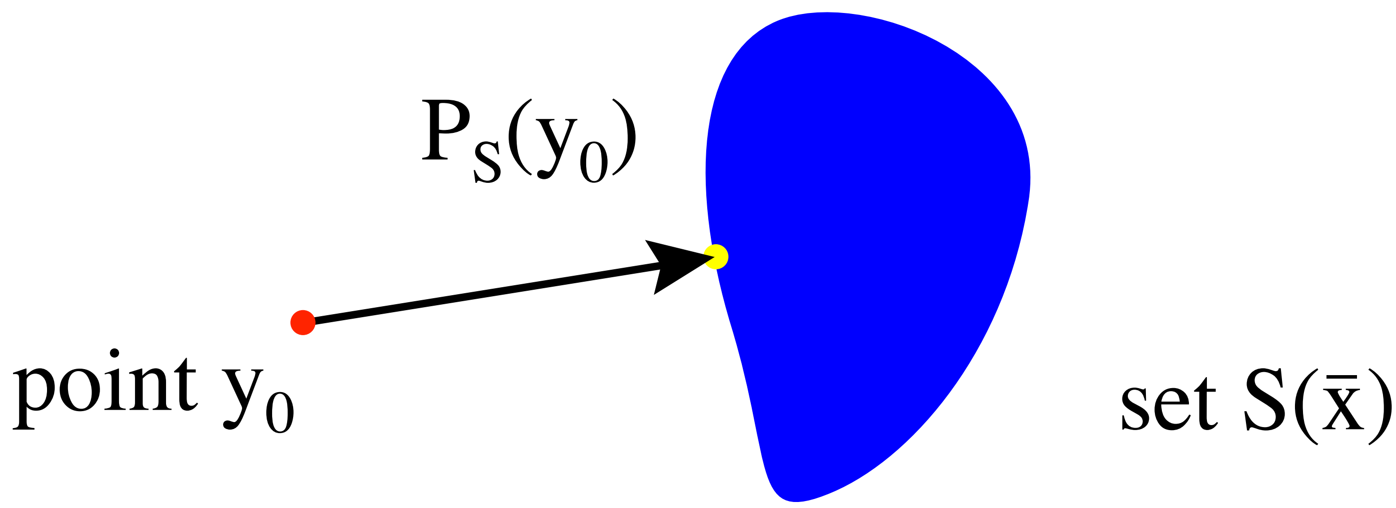


Minimal size of backward error is

$$\mu(\bar{x}) = \operatorname{dist}(y_0, \underbrace{\{y' : F(y', \bar{x}) = 0\}}_{S(\bar{x})})$$

all data compatible with $\bar{x}$

Metric Projection



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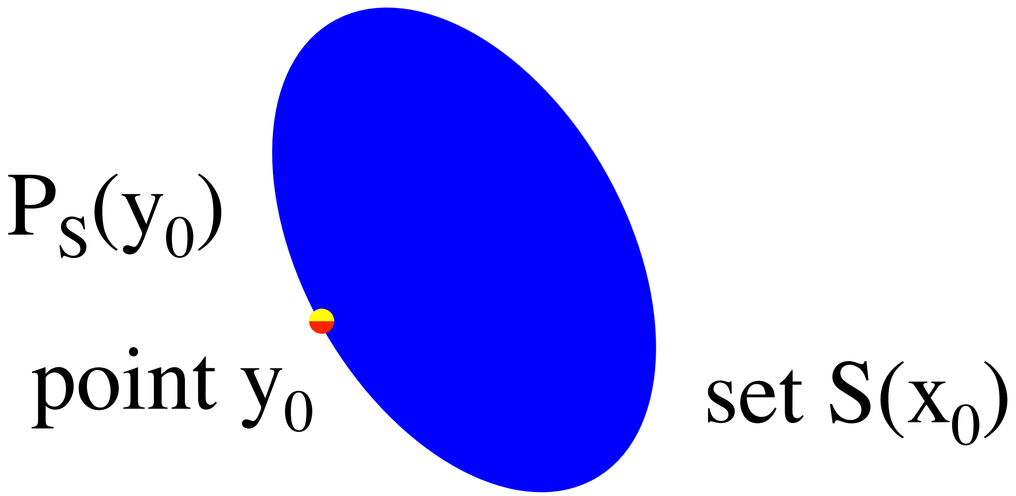
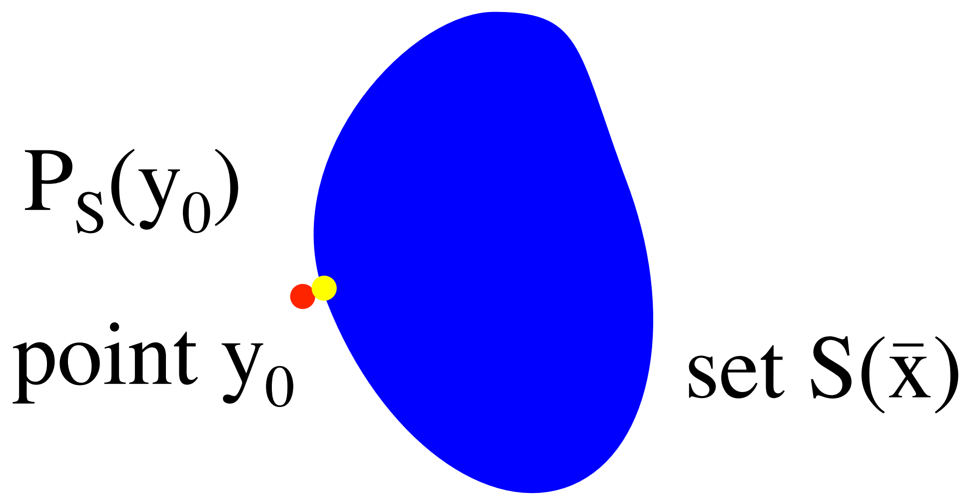


Minimal size of backward error is

$$\mu(\bar{x}) = \operatorname{dist}\Big(y_0, \underbrace{\{y' : F(y', \bar{x}) = 0\}}_{S(x_0)}\Big)$$

all data compatible with x_0

Metric Projection



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$$\mu(\bar{x}) = \operatorname{dist}\left(y_0, \underbrace{\{y' : F(y', \bar{x}) = 0\}}_{S(\bar{x})}\right)$$

all data compatible with $\bar{x}$

Summary of Abstract Formulation

Minimal size of backward error is a distance

$$\mu(\bar{x}) = \min_{y' \in \mathcal{S}(\bar{x})} \|y' - y_0\|$$

$$\mathcal{S}(\bar{x}) = \{y' : F(y', \bar{x}) = 0\}$$

- ⋆ The set $\mathcal{S}(\bar{x})$ is subject to change.
- ⋆ The point y_0 is not subject to change.
- ⋆ $\bar{x} \approx x_0$, which places $\mathcal{S}(\bar{x})$ near y_0.
- ⋆ The true solution x_0 is unknown.

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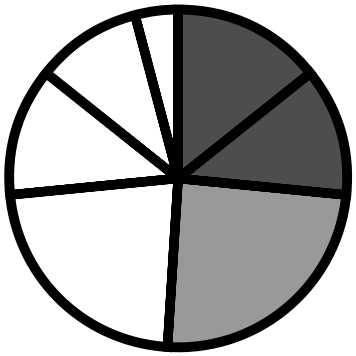

Differentiation with respect to $\boldsymbol{y_0}$ (not $\boldsymbol{\mathcal{S}}$) for

$$\left\{\begin{matrix} \operatorname{dist}(\boldsymbol{y_0}, \boldsymbol{\mathcal{S}}) \\ \boldsymbol{P_{\mathcal{S}}}(\boldsymbol{y_0}) \end{matrix}\right\} \times \left\{\begin{matrix} \boldsymbol{y_0} \in \boldsymbol{\mathcal{S}} \\ \boldsymbol{y_0} \notin \boldsymbol{\mathcal{S}} \end{matrix}\right\} \times \left\{\begin{matrix} \text{convex } \boldsymbol{\mathcal{S}} \\ \text{unconvex} \end{matrix}\right\} \times \cdots$$

$$\left\{\begin{matrix} \text{Hilbert space} \\ \text{Banach space} \end{matrix}\right\} \times \left\{\begin{matrix} \text{finite dimensional} \\ \infty \text{ dimensional space} \end{matrix}\right\} = \boldsymbol{2^5}$$

★ Basic negative result: $\boldsymbol{P_{\mathcal{S}}}(\boldsymbol{y_0})$ need not be directionally differentiable everywhere in $\mathbb{E}^{\mathbf{2}}$ for convex $\boldsymbol{\mathcal{S}}$.

— [Kruskal, 69] [Shapiro, 94]

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Differentiation with respect to $\boldsymbol{y}_0$ (not $\boldsymbol{\mathcal{S}}$) for

$$\left\{\begin{array}{c} \operatorname{dist}(\boldsymbol{y}_0, \boldsymbol{\mathcal{S}}) \\ \boldsymbol{P}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{y}_0) \end{array}\right\} \times \left\{\begin{array}{c} \boldsymbol{y}_0 \in \boldsymbol{\mathcal{S}} \\ \boldsymbol{y}_0 \notin \boldsymbol{\mathcal{S}} \end{array}\right\} \times \left\{\begin{array}{c} \text{convex } \boldsymbol{\mathcal{S}} \\ \text{unconvex} \end{array}\right\} \times \cdots$$

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★ Basic positive result: $\boldsymbol{P}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{y}_0)$ is directionally differentiable everywhere at boundary of convex $\boldsymbol{\mathcal{S}}$ in Hilbert spaces.

— [Zarantonello, 71]

Differentiation with respect to y_0

1. Differentiability at internal points $y_0 \in \text{bd}(S)$:
   (a) Convex S:
   Hilbert spaces: $P_S(y_0)$ is directionally differentiable always. [Zarantonello, 71]
   (b) Arbitrary S:
   Finite dimensional Banach spaces: y_0 and S have been characterized for which $P_S(y_0)$ is directionally differentiable [Shapiro, 87]

2. Differentiability at external points $y_0 \notin S$:
   (a) Convex S:
   Banach spaces: $\text{dist}(y_0, S)$ is continuously differentiable in spaces with differentiable norms [Holmes, 73]
   (b) Arbitrary S:
   Hilbert spaces: sets have been classified that have uniform envelopes where $\text{dist}(y_0, S)$ is continuously differentiable [Clarke et al., 95]

$$\phi(x) = \min_{y \,:\, G(y,x) \in \mathcal{K}} g(y,x)$$

"It is difficult to investigate the sensitivity of an optimal value whose feasible set is subject to change . . . "

[Bonnans and Shapiro, 98, 00]

Used for:

- sensitivity analysis [Fiacco and Ghaemi, 82]
- finding optimality conditions
- establishing the convergence of algorithms

Theory assumes continuous 2$^{\text{nd}}$ derivatives for both constraint and objective functions.

★ In Hilbert spaces [B&S, 00]

$$\lim_{t \to 0^+} \frac{\mu(x_0 + t\Delta x)}{t} = \min_{\Delta y : J_1 \Delta y + J_2 \Delta x = 0} \|\Delta y\|_2$$

- If F has continuous second derivatives,
- $[J_1, J_2] = \mathcal{D}F(y_0, x_0)$ has full row rank.

First Order Approximation to $\mu(\bar{x})$

Use the result of B&S to estimate

$$\mu(\bar{x}) = \min_{\{y' \,:\, F(y', \bar{x}) = 0\}} \|y' - y_0\|_2$$

The directional derivative gives to 1^{st} order

$$\mu(x_0 + \Delta x) = \min_{\Delta y \,:\, J_1 \Delta y + J_2 \Delta x = 0} \|\Delta y\|_2 + \mathcal{O}(\|\Delta x\|^2)$$

where $[J_1, J_2] = \mathcal{D}F(y_0, x_0)$.

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Drawback: Only 2-Norm Backward Error

Use the result of B&S to estimate

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Drawback: Remainder not Proved Uniform

Use the result of B&S to estimate

$$\mu(\bar{x}) = \min_{\{y' \,:\, F(y', \bar{x}) = 0\}} \|y' - y_0\|_2$$

The directional derivative gives to 1^{st} order

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where $[J_1, J_2] = \mathcal{D}F(y_0, x_0)$.

Summary of Prior Work

Regarding (Directional) Derivatives of μ . . .

1. Pure Math

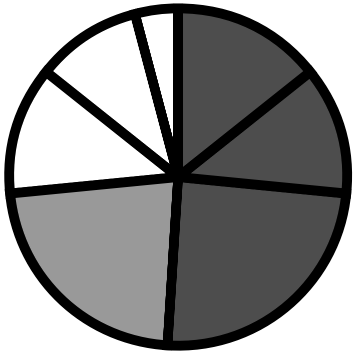
   Only perturbs y_0 — inapplicable to varying $\mathcal{S}$.

2. Optimization Theory

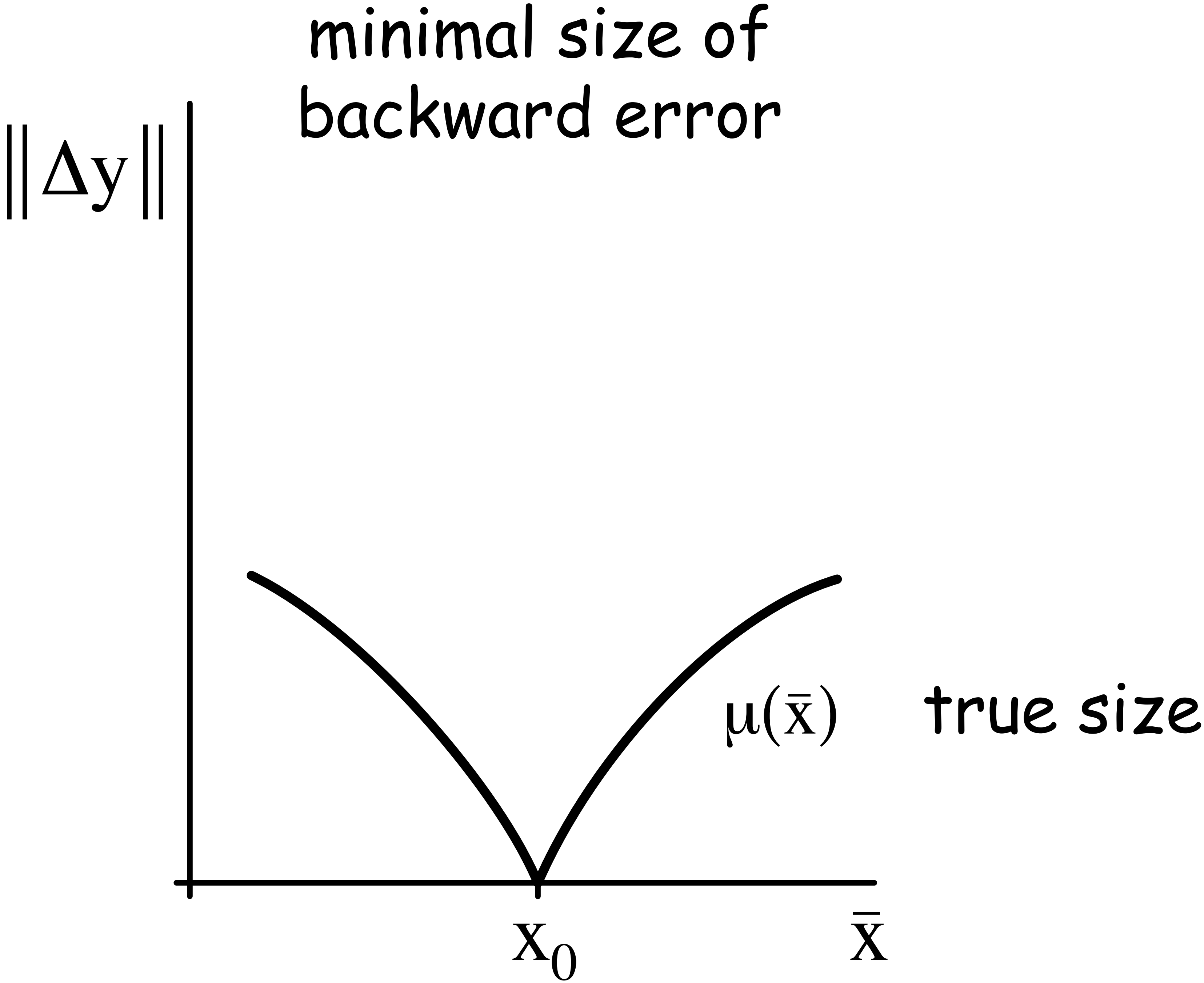
   Studies perturbations to $\mathcal{S}$. ✓

   - Shows derivative linearizes the constraint.
   - Requires 2$^{\text{nd}}$ order differentiability. :-(
     - So only for 2-norm — drawback.
   - Remainder not uniform in direction — serious.
   - Formula needs Δx & x_0 — show stopper.

Outline



How to Estimate $\mu(\bar{x})$?



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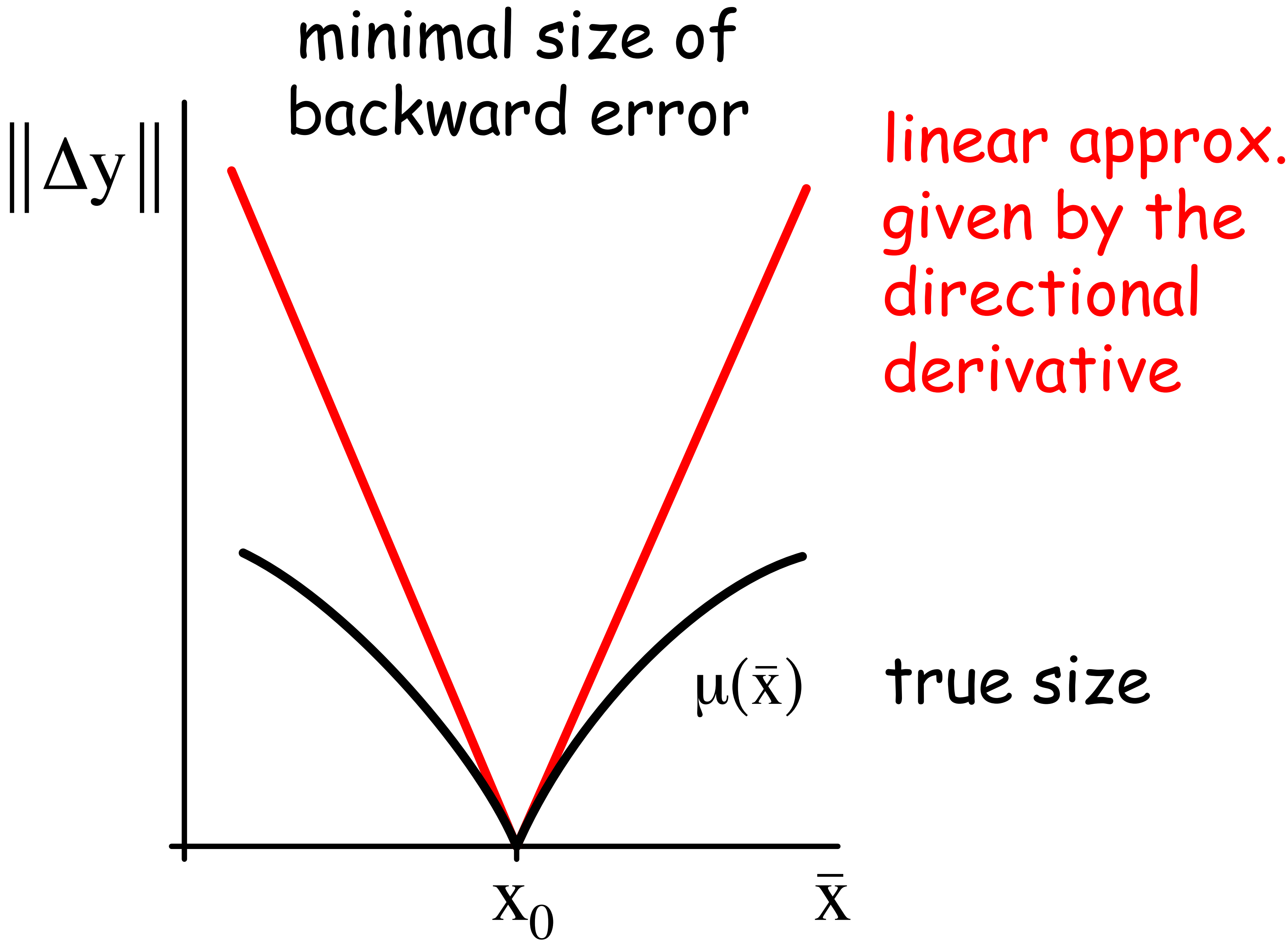

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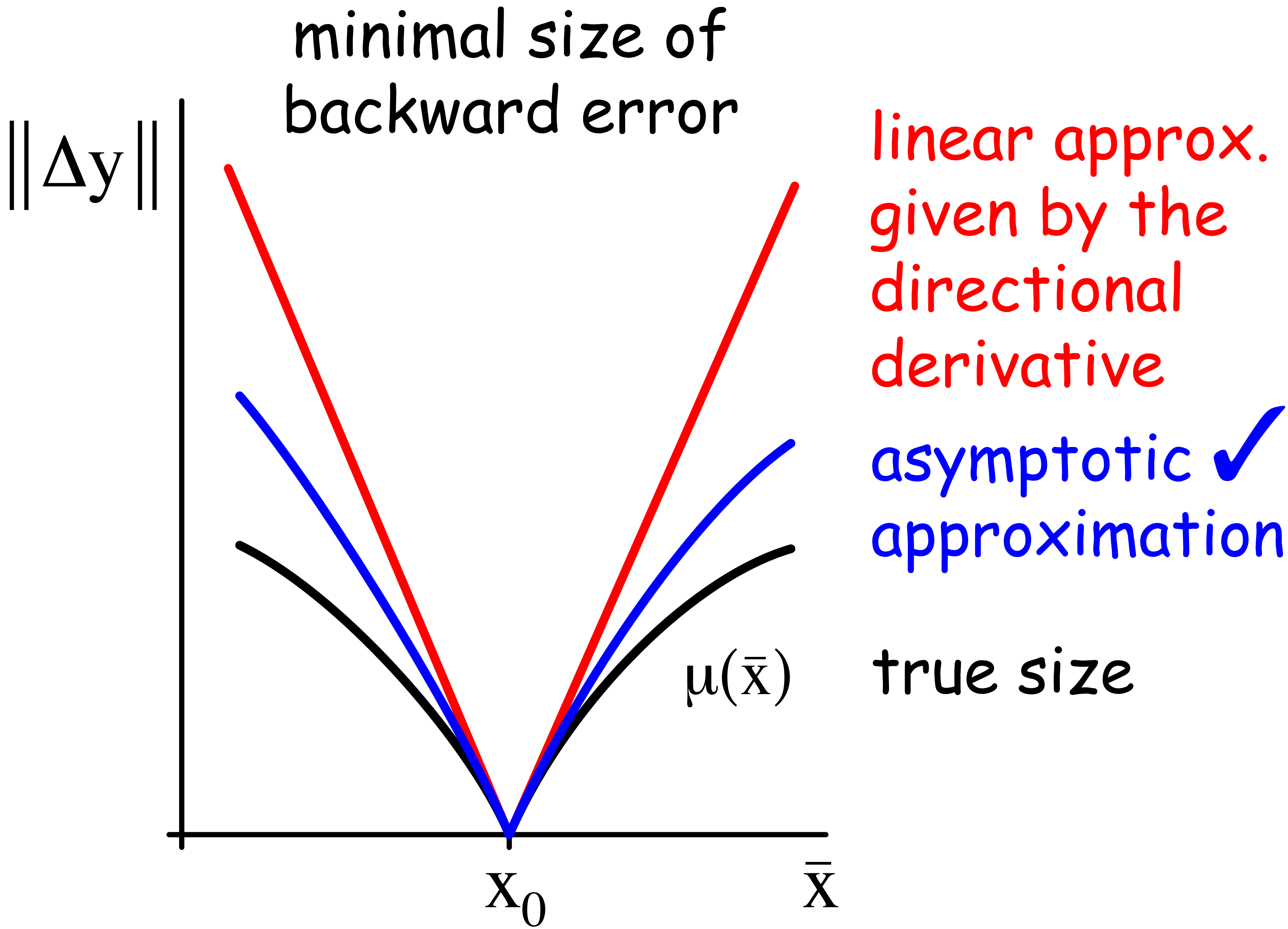

If real-valued functions f and g satisfy

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = 1$$

then $f \sim g$ at x_0

f asymptotically approximates g, or

f and g are asymptotically equivalent.

I.e., $\forall\, \epsilon > 0 \;\; \exists\, \delta > 0$ so that $\|x - x_0\| < \delta \;\Rightarrow$

$$(1 - \epsilon)\, g(x) \leq f(x) \leq (1 + \epsilon)\, g(x)$$

Asymptotic Implies Differential

If functions f and g satisfy

$$\mathcal{D}(f - g)(x_0) = 0$$

then at x_0

f differentially approximates g, or

f and g are differentially equivalent.

★ If f, g are asymptotically equivalent at x_0 and if one of f or g is Lipshitz continuous at x_0, then they are differentially equivalent.

3 Linearizations May Estimate $\mu(\bar{x})$

In the minimal size of backward error

$$\mu(\bar{x}) = \min_{\{y' \,:\, F(y', \bar{x}) = 0\}} \|y' - y_0\|$$

there are many ways to approximate the constraint

$$F(y', \bar{x}) = 0$$

1. $\mathcal{D}_1 F(y_0, \bar{x})\, \Delta y + F(y_0, \bar{x}) = 0$
2. $\mathcal{D}_1 F(y_0, x_0)\, \Delta y + F(y_0, \bar{x}) = 0$
3. $\mathcal{D}_1 F(y_0, x_0)\, \Delta y + \mathcal{D}_2 F(y_0, x_0)\, \Delta x = 0$

<u>Theorem</u>: For residual function F and data y_0,

1. if F is continuously Fréchet differentiable,
2. if there is a solution x_0, i.e. $F(y_0, x_0) = 0$
3. if $\mathcal{D}_1 F(y_0, x_0)$ has full row rank, then

then the minimal size of the backward error

$$\mu(\bar{x}) = \min_{y' \,:\, F(y', \bar{x}) = 0} \|y' - y_0\|$$

is asymptotically estimated by replacing the constraint with the first 2 approximations.

Dual of a Linearly Constrained Problem

If $\mathcal{L} : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ maps one space onto another, then

$$\min_{u\,:\,\mathcal{L}u = h} \|u\| = \max_{g \in (\mathbb{R}^p)^*} \frac{g(h)}{\|\mathcal{L}^* g\|^*} = \|h\|_{\mathcal{L}}$$

For full row rank matrices $\mathbf{J}$ and 2-norms,

$$\min_{u\,:\,\mathbf{J}u = h} \|u\|_2 = \max_{g} \frac{g^t h}{\|\mathbf{J}^t g\|_2} = \|\mathbf{J}^\dagger h\|_2$$

Theorem's 1$^{\text{st}}$ Estimate is Computable

$\boldsymbol{h} = \boldsymbol{F}(\boldsymbol{y_0}, \bar{\boldsymbol{x}})$ is the residual of the problem

$\mathbf{J} = \mathcal{D}_1 \boldsymbol{F}(\boldsymbol{y_0}, \bar{\boldsymbol{x}})$ Jacobian of residual w.r.t. data

$$\boldsymbol{\mu}^{(1)}(\bar{\boldsymbol{x}}) \sim \min_{\boldsymbol{\Delta y} : \mathbf{J}\,\boldsymbol{\Delta y} = \boldsymbol{h}} \overbrace{\|\boldsymbol{\Delta y}\| \;=\; \|\mathbf{J}^{\dagger}\boldsymbol{h}\|_{\mathbf{2}}}^{\text{if } \mathbf{2} \text{ norms}}$$

★ Nothings depends on, $\boldsymbol{x_0}$, the true solution.

★ The estimate can be evaluated.

Theorem's 2nd Estimate is Canonical

$h = F(y_0, \bar{x})$ is the residual of the problem

$\mathbf{J} = \mathcal{D}_1 F(y_0, x_0)$ Jacobian of residual w.r.t. data

$$\mu^{(2)}(\bar{x}) \sim \max_{g} \frac{g^t h}{\|\mathbf{J}^t g\|^*} = \|h\|_{\mathbf{J}} \overbrace{= \ \|\mathbf{J}^\dagger h\|_2}^{\text{if } \mathbf{2} \text{ norms}}$$

★ The minimal size of the backward error is asymptotically a norm of the residual.

★ The norm is unique.

Linearized equations give asymptotic estimates for the minimal size (in any norm) of the backward error of numerical problem.

1^{st} **estimate** can be computed for 2-norms by solving a large, sparse LLS problem.

2^{nd} **estimate** shows the minimal size of the backward error is uniquely determined as a norm of the equations' residual.

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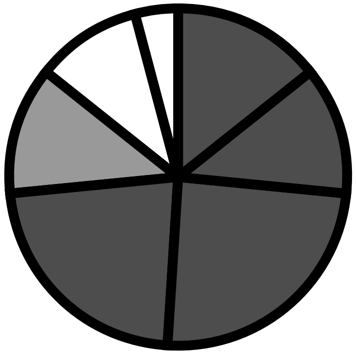

$x_0 = \arg\min_x \|b - A_0 x\|_2$

Find easily computable statistics that are both necessary and sufficient for the stability of a least squares solution. — [Stewart (and Wilkinson), 77]

Exactly minimal size — [Waldén, Karlson, Sun, 95]

$$\mu(\bar{x}) = \min_{\Delta A} \|\Delta A\|_F =$$

$$= \sqrt{\frac{\|\bar{r}\|_2^2}{\|\bar{x}\|_2^2} + \min_i \left\{ 0,\ \lambda_i \left(A_0 A_0^t - \frac{\bar{r}\,\bar{r}^{\,t}}{\|\bar{x}\|_2^2} \right) \right\}}$$

where $\bar{r} = b - A_0 \bar{x}$

Step 1: Check the Theorem's Hypotheses

1. Continuously differentiable equations

$$F(A, x) = A^t(b - Ax)$$

2. Any A_0 has at least one solution, x_0

3. $J = \mathcal{D}_1 F(A_0, x_0) =$

$$\begin{bmatrix} e_1 r_0^t & e_2 r_0^t & \cdots & e_n r_0^t \end{bmatrix} - \begin{bmatrix} x_1 A_0^t & x_2 A_0^t & \cdots & x_n A_0^t \end{bmatrix}$$

where $r_0 = b - A_0 x_0$ is the true residual. Since $A_0^t r_0 = 0$,

$$J J^t \;=\; \|r_0\|_2^2 \, I + \|x_0\|_2^2 \, A_0^t A_0$$

so J has full row rank provided $r_0 \neq 0$.

Step 2: Form the 2^{nd} Estimate

The $\mathbf{2^{nd}}$ asymptotic estimate is

$$\begin{aligned} \boldsymbol{\mu^{(2)}(\bar{x})} &= \| \boldsymbol{J^{\dagger} F(A_0, \bar{x})} \|_2 \\ &= \| (\boldsymbol{J J^t})^{-1/2} \boldsymbol{A_0^t \bar{r}} \|_2 \\ &= \| \left(\|\boldsymbol{r_0}\|_2^2 \boldsymbol{I} + \|\boldsymbol{x_0}\|_2^2 \boldsymbol{A_0^t A_0} \right)^{-1/2} \boldsymbol{A_0^t \bar{r}} \|_2 \end{aligned}$$

where

$\boldsymbol{r_0 = b - A_0 x_0}$ true least squares residual

$\boldsymbol{\bar{r} = b - A_0 \bar{x}}$ residual of computed $\boldsymbol{\bar{x}}$

Step 3: Replace $x_0 \longrightarrow \bar{x}$

$$\mu^{(2)}(\bar{x}) = \| \left(\|r_0\|_2^2\, I + \|x_0\|_2^2\, A_0^t A_0\right)^{-1/2} A_0^t\, \bar{r}\, \|_2$$

$$\tilde{\mu}(\bar{x}) = \| \left(\|\bar{r}\|_2^2\, I + \|\bar{x}\|_2^2\, A_0^t A_0\right)^{-1/2} A_0^t\, \bar{r}\, \|_2$$

This too is asymptotic

$$\lim_{\bar{x}\longrightarrow x_0} \frac{\tilde{\mu}(\bar{x})}{\mu(\bar{x})} = \lim_{\bar{x}\longrightarrow x_0} \frac{\tilde{\mu}(\bar{x})}{\mu^{(2)}(\bar{x})} \frac{\mu^{(2)}(\bar{x})}{\mu(\bar{x})} = 1$$

For comparison, the exact value is

$$\mu(\bar{x}) = \sqrt{\frac{\|\bar{r}\|_2^2}{\|\bar{x}\|_2^2} + \min\left\{0,\ \lambda_{\min}\left(A_0 A_0^t - \frac{\bar{r}\,\bar{r}^{\,t}}{\|\bar{x}\|_2^2}\right)\right\}}$$

$\tilde{\mu}(\bar{x})$ Has Appeared Before

Both Kalson and Waldén, and Gu used formulas equivalent to $\tilde{\mu}(\bar{x})$ to derive other bounds on $\mu(\bar{x})$. Their intermediate results include

- [Karlson and Waldén, 97]

$$\frac{\tilde{\mu}(\bar{x})}{\mu(\bar{x})} \le \frac{2+\sqrt{2}}{2} \approx 1.707$$

- when A_0 has full column rank [Gu, 99]

$$1 \approx \frac{\|r_0\|_2}{\|\bar{r}\|_2} \le \frac{\tilde{\mu}(\bar{x})}{\mu(\bar{x})} \le \frac{\sqrt{5}+1}{2} \approx 1.618$$

- so roughly $\mu(\bar{x}) \le \tilde{\mu}(\bar{x}) \le 2\,\mu(\bar{x})$

Summary of Application to LLS

- The perturbation theorem can be used to easily derive computable asymptotic estimates for the minimal size of the the backward error of LLS,

$$\tilde{\mu}(\bar{x}) = \| \left(\|\bar{r}\|_2^2 \, I + \|\bar{x}\|_2^2 \, A_0^t A_0\right)^{-1/2} A_0^t \, \bar{r} \, \|_2$$

provided $r_0 \neq 0$ (no restriction on rank of A_0).

- Other results show this is boundedly near $\mu(\bar{x})$.
- And other differential results show any estimate must be of this form.

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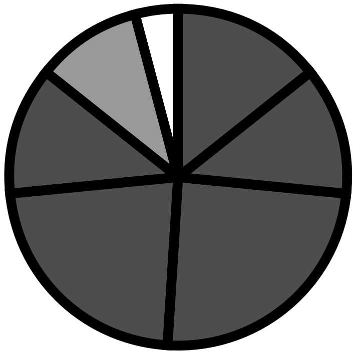


Evaluating the Estimate $\tilde{\boldsymbol{\mu}}(\bar{\boldsymbol{x}})$

Joseph F. Grcar
Lawrence Berkeley Laboratory

Michael Saunders   Zheng Su
Stanford University

Householder XVI – May 26, 2005

If a matrix K has full column rank, then the orthogonal projection into col(K) satisfies

$$\|\mathcal{P}_K v\|_2 \;=\; \left\| \left[(K^t K)^{-1/2} K^t \right] v \right\|_2$$

Notice that $\tilde{\mu}(\bar{x}) = \|\mathcal{P}_K v\|_2$ for the choices

$$K = \begin{bmatrix} A_0 \\ \frac{\|\bar{r}\|_2}{\|\bar{x}\|_2}\, I \end{bmatrix} \quad \text{and} \quad v = \frac{1}{\|\bar{x}\|_2} \begin{bmatrix} \bar{r} \\ 0 \end{bmatrix}$$

Calculate Estimate Given $A_0 = QR$

Since $A_0 = QR$ is available, use

$$K' = \begin{bmatrix} R \\ 0 \\ \frac{\|\bar{r}\|_2}{\|\bar{x}\|_2} I \end{bmatrix} \quad \text{and} \quad v' = \frac{1}{\|\bar{x}\|_2} \begin{bmatrix} Q^t \bar{r} \\ \cdots \\ 0 \end{bmatrix}$$

The zero rows can be discarded, leaving

$$K'' = \begin{bmatrix} R \\ \frac{\|\bar{r}\|_2}{\|\bar{x}\|_2} I \end{bmatrix} \quad \text{and} \quad v'' = \frac{1}{\|\bar{x}\|_2} \begin{bmatrix} Q^t \bar{r} \\ 0 \end{bmatrix}$$

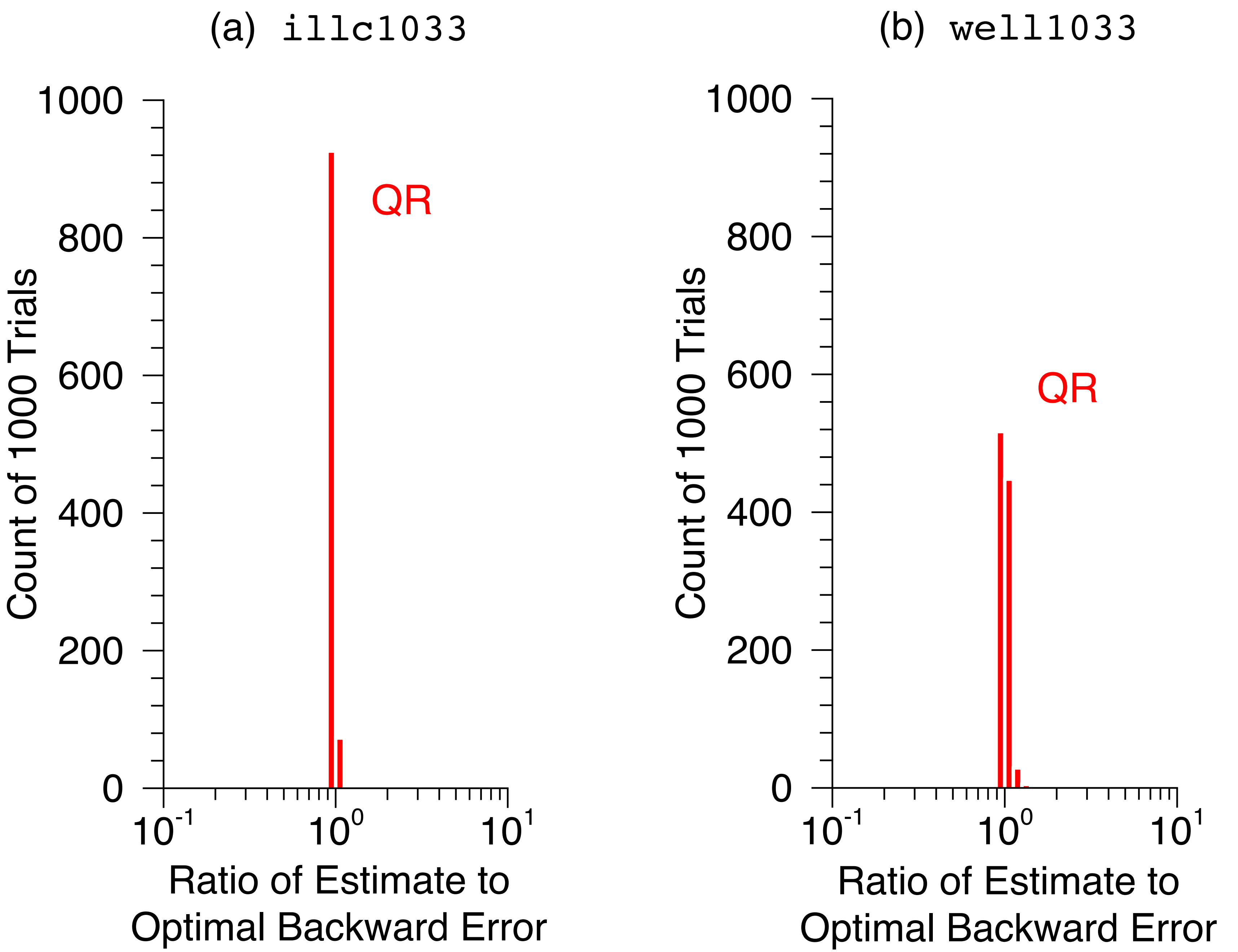
Calculate Given $A_0 = QR$, cont.

It is easy to factor $K'' = Q_K R_K$ using plane rotations [Karlson and Waldén, 97].

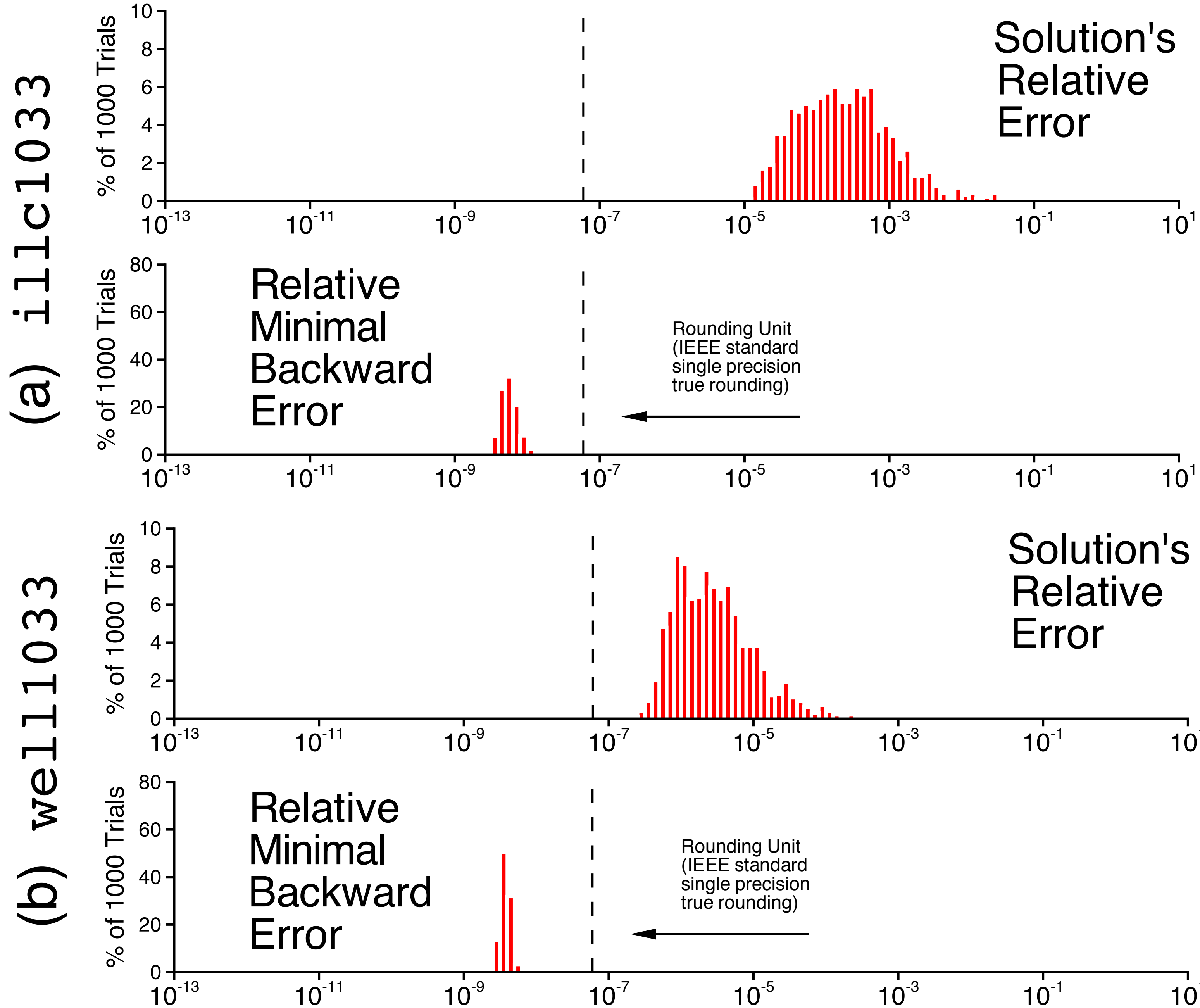
$$\tilde{\mu}(\bar{x}) = \|\mathcal{P}_K v\|_2 = \|\mathcal{P}_{K''} v''\|_2 = \frac{\|Q^t_{K''}\, Q^t_A\, \bar{r}\,\|_2}{\|\bar{x}\|_2}$$

solve LLS by Householder QR	$2mn^2$
form $Q^t_A \bar{r}$	$4mn$
apply $Q^t_{K''}$ to $Q^t_A\, \bar{r}$	$\frac{8}{3}n^3$
finish evaluating $\tilde{\mu}(\bar{x})$	$2n$

Histograms of Ratio $\tilde{\mu}(\bar{x})/\mu(\bar{x})$



Relative Forward and Backward Errors



1. Accuracy criterion. [von Neumann, 47]
   If the initial data have error $\geq \mu(\bar{x})$,
   then $\bar{x}$ solves the problem,
   to the extent the problem is known.

2. Backward stability estimation.
   An $\bar{x}$ with a small $\mu(\bar{x})$ is backward stable.

3. Test new algorithms.
   Explore backward stability without having to do an inverse rounding error analysis.

Outline

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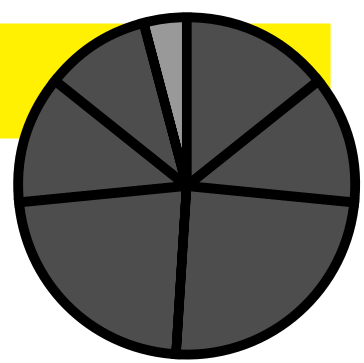

Conclusions

1. Perturbation theory of metric projections provides asymptotic estimates for optimal backward errors.

2. The estimates can be applied to practical problems such as linear least squares.

3. The estimates for LLS are inexpensive and accurate, answering Stewart and Wilkinson’s question.

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How to Use the 1st Estimate

Suppose a calculation (program)

1. has inputs (data) $\boldsymbol{y_0}$
2. has outputs (computed solution) $\bar{\boldsymbol{x}}$
3. is meant to solve equations $\boldsymbol{h} = \boldsymbol{F}(\boldsymbol{y}, \boldsymbol{x}) = \mathbf{0}$

Do this

1. form the Jacobian matrix, $\mathbf{J}$, of $\boldsymbol{F}$ w.r.t. $\boldsymbol{y}$
2. evaluate $\mathbf{J}$ and $\boldsymbol{h}$ at $\boldsymbol{y} = \boldsymbol{y_0}$ and $\boldsymbol{x} = \bar{\boldsymbol{x}}$
3. use QR or SVD to evaluate $\|\mathbf{J}^{\dagger}\boldsymbol{h}\|_{\mathbf{2}} \sim \boldsymbol{\mu}(\bar{\boldsymbol{x}})$

Example of Using the 1st Estimate

Saddle point problem

$$\begin{bmatrix} A & B^t \\ B & C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Want backward error to honor the structure

- not separate perturbations to B and B^t
- not perturbations to zeroes in A, B, C

Saddle point problem

$$F(y_0, \bar{x}) = \begin{bmatrix} A & B^t \\ B & C \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Want backward error to honor the structure

- inputs $y_0 =$ (entries of A, B, C, b_1, b_2)
- outputs $\bar{x} = (\bar{x}_1, \bar{x}_2)$
- $\mathbf{J} =$